On the Hyperbolic Gluing Equations and Representations of Fundamental Groups of Closed 3-Manifolds

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Abstract

We show that for a representation of the fundamental group of a triangulated closed 3-manifold (not necessarily hyperbolic) into $PSL_2(\mathbb{C})$ so that any edge loop has non-trivial image under the representation, there exist uncountably many solutions to the hyperbolic gluing equation whose associated representations are conjugate to the given representation, and whose volumes are equal to the volume of the given representation.

1 Introduction

In [10], Thurston introduced a system of algebraic equations –called the *hyperbolic gluing equations* for constructing hyperbolic metrics on orientable 3-manifolds with torus cusps. He used solutions to the hyperbolic gluing equations to produce a complete hyperbolic metric on the figure-eight knot complement in the early stages of formulating his geometrization conjecture. On a closed, oriented, triangulated 3-manifold M, the hyperbolic gluing equations can be defined in the same way: We assign to each edge of each oriented tetrahedron in the triangulation a shape parameter $z \in \mathbb{C} \setminus \{0, 1\}$ such that

(a) opposite edges of each tetrahedron have the same shape parameter;

(b) the three shape parameters assigned to three pairs of opposite edges in each tetrahedron are z, $\frac{1}{1-z}$ and $1 - \frac{1}{z}$ subject to an orientation convention; and

(c) for each edge e in M, if $z_1, ..., z_k$ are shape parameters assigned to the edges sharing e as an edge, then we have

$$\prod_{i=1}^{k} z_i = 1. \tag{1}$$

The equations (1) are termed *the hyperbolic gluing equations* , and the set of all solutions is the *parameter space* $\mathcal{P}(M)$. The space $\mathcal{P}(M)$ depends on the triangulation of M. Given any $Z \in \mathcal{P}(M)$, the *associated representation*, denoted ρ_Z, is defined by Yoshida in [11]; and the *volume* of Z, denoted $Vol(Z)$, is well-defined using the Lobachevsky-Milnor formula.

In our joint work with F.Luo and S.Tillmann [5], we were able to show the hyperbolic structure on a closed, oriented, triangulated *hyperbolic* 3-manifold can be constructed from a solution to the hyperbolic gluing equation using any triangulation with essential edges. An edge in $\mathcal{T}$ is termed *essential* if it is not a null homotopic loop in M. This is clear the case if it has distinct end-point, but we allow the triangulation of M to be semi-simplicial (or singular), so that some or all edges may be loops in M. It is well known that a closed 3-manifold M is hyperbolic if and only if there exists a discrete and faithful representation $\rho\colon\ \pi_1(M) \to PSL_2(\mathbb{C})$ of the fundamental group of M into $PSL_2(\mathbb{C})$, and the main

result of [5] is to construct a family of solutions to the hyperbolic gluing equation all of whose associated representations are discrete and faithful. The proof makes a crucial use of Thurston's spinning construction and a volume rigidity theorem attributed by Dunfield to Thurston, Gromov and Goldman.

Our main observation in the present paper is that the constrain that the 3-manifold M is hyperbolic, or equivalently the representation $\rho\colon \pi_1(M) \to PSL_2(\mathbb{C})$ is discrete and faithful, could be removed, and we have

Theorem 1.1 *Let M be an oriented, closed 3-manifold, $\mathcal{T}$ be a triangulation of M so that each edge e in $\mathcal{T}$ is essential, and $\rho\colon \pi_1(M) \to PSL_2(\mathbb{C})$ be a representation of the fundamental group of M into $PSL_2(\mathbb{C})$ so that*

$$\rho([e]) \neq 1, \text{ for all loop } e \text{ in } \mathcal{T}. \tag{2}$$

Then

1. *there exist uncountably many solutions Z_ρ to the hyperbolic gluing equation;*
2. *the associated representation ρ_{Z_ρ} is conjugate to ρ; and*
3. *$Vol(Z_\rho) = Vol(\rho)$.*

We note that, in the special case that $\mathcal{T}$ is simplicial, i.e., every 3-simplex in $\mathcal{T}$ has distinct vertices, every representation $\rho\colon \pi_1(M) \to PSL_2(\mathbb{C})$ satisfies condition (2); and in the case that $\rho\colon \pi_1(M) \to PSL_2(\mathbb{C})$ is discrete and faithful, Theorem 1.1 implies Theorem 1.1 of [5] as a special case.

The present paper is organized as follows. In section 2, some basic definitions on hyperbolic geometry are reviewed. Theorem 1.1 is proven in section 3 using the spinning construction summarized in [5] and a theorem of Luo on the continuous extension of the volume function [3].

2 The parameter space

2.1 The hyperbolic gluing equation and the volume of solutions

If σ is an oriented 3-simplex with edges from one vertex labeled by e_1, e_2 and e_3 so that the opposite edges have the same labeling, then the cyclic order of e_1, e_2 and e_3 viewed from each vertex depends only on the orientation of the tetrahedron, i.e. is independent of the choice of the vertices. Note that each pair of opposite edges e_i corresponds to a normal isotopy class of quadrilateral (*normal quadrilateral* for short) q_i in σ so that $q_1 \to q_2 \to q_3 \to q_1$ is the cyclic order induced by the cyclic order on the edges from a vertex. To define hyperbolic gluing equation, we need the following notation. Let e be an edge in $\mathcal{T}$, and q be a normal quadrilateral in σ. The index $i(q,e)$ is the integer 0, 1 or 2 defined as follows. $i(q,e) = 0$ if e is not an edge of σ. $i(q,e) = 1$ if e is the only edge in σ facing q and $i(q,e) = 2$ if e are the two edges in σ facing q. Let Q be the set of normal quadrilaterals in $\mathcal{T}$, we have

Definition 2.1 *Suppose $(M, \mathcal{T})$ is a triangulated oriented close 3-manifold. Thy hyperbolic gluing equation is defined for $Z = (z_q) \in (\mathbb{C} \setminus \{0,1\})^Q$ so that*

(a) for each edge e in $\mathcal{T}$,

$$\prod_{q \in Q} z_q^{i(q,e)} = 1,$$

and

(b) if σ is a 3-simplex in $\mathcal{T}$, and $q_1 \to q_2 \to q_3 \to q_1$ is the cyclic order of normal quadrilaterals in σ, then

$$z_{q_{i+1}} = \frac{1}{1 - z_{q_i}},$$

where q_{3+1} is understood to be q_1.

The set of all solutions to the hyperbolic gluing equation is call the parameter space, and is denoted by $\mathcal{P}(M)$.

Let $z_\sigma = (z_{q_1}, z_{q_2}, z_{q_3})$ be the complex numbers assigned to q_i, $i \in \{1, 2, 3\}$, then we have

Definition 2.2 *The volume of z_σ is defined to be the sum of the Lobachevsky functions*

$$\begin{aligned} Vol(z_\sigma) &= \sum_{i=1}^{3} \Lambda(\arg(z_{q_i})) \\ &\doteq \sum_{i=1}^{3} (- \int_0^{\arg(z_{q_i})} \ln |2 \sin t| dt); \end{aligned}$$

and the volume of $Z = (z_\sigma) \in \mathcal{P}(M)$ is defined by

$$Vol(Z) = \sum_{\sigma \in T} Vol(z_\sigma).$$

2.2 The shape parameters of an ideal tetrahedron

Let $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup S^2_\infty$ be the compactification of $\mathbb{H}^3$, where S^2_∞ is the *sphere at infinity.* We have

Definition 2.3 *Let σ be an ideal tetrahedron in $\overline{\mathbb{H}^3}$ with vertices $\{v_i\} \subset S^2_\infty$, $i \in \{1, ..., 4\}$, and e_{ij} be the edge from v_i to v_j. Identifying S^2_∞ with $\mathbb{C} \cup \{\infty\}$, the shape parameter of σ at e_{ij} is defined by the following cross-ratio*

$$\begin{aligned} z_{ij} \doteq &(v_i, v_j; v_k, v_l) \\ = &\frac{v_i - v_k}{v_i - v_l} \cdot \frac{v_j - v_l}{v_j - v_k} \end{aligned}$$

where (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$.

A direct cross-ratio calculation shows the following well known

Proposition 2.4 *1. For all $\{i, j\}, \{k, l\} \subset \{1, ..., 4\}$, $i, j \neq k, l$,*

$$z_{ij} = z_{kl},$$

i.e., opposite edges share the same shape parameter, so we can denote the shape parameter of σ at e_{ij} and e_{kl} by z_q, where q is the normal quadrilateral facing e_{ij} and e_{kl}, and

2. if $q_1 \to q_2 \to q_3 \to q_1$ is the cyclic order of normal quadrilaterals in σ, then

$$z_{q_{i+1}} = \frac{1}{1 - z_{q_i}},$$

where q_{3+1} is understood to be q_1.

For an ideal tetrahedron σ with shape parameters z_{q_1}, z_{q_2} and z_{q_3}, the hyperbolic volume is calculated by Milnor as

$$Vol_{\mathbb{H}^3}(\sigma) = \sum_{i=1}^{3} \Lambda(\arg(z_{q_i})).$$

We call an ideal tetrahedron $\sigma \subset \overline{\mathbb{H}^3}$ *flat* if it lies in a totally geodesic plan. When σ is flat, we have that $\{z_{q_i}\}$ are real numbers and $Vol(\sigma) = 0$.

2.3 The associated representation

Given a solution $Z \in \mathcal{P}(M)$ to the hyperbolic gluing equation for a triangulated cusped 3-manifold $(M, \mathcal{T})$ with essential edges, the associated representation $\rho_Z\colon \pi_1(M) \to PSL_2(\mathbb{C})$ was described by Yoshida [11] via constructing the pseudo developing maps. For a closed triangulated 3-manifold with essential edges, the construction of the associated representation is essentially the same as Yoshida's. Namely, for each $Z \in \mathcal{P}(M)$, there is a continuous map $D_Z\colon \widetilde{M} \to \overline{\mathbb{H}^3}$ taking 3-simplices in $\widetilde{\mathcal{T}}$ to ideal straight simplices, and a representation ρ_Z which makes it invariant. This construction is described in details in Section 4.5, [5].

3 The existence of solutions to the hyperbolic equation

3.1 The proof of 1. of Theorem 1.1

Proof Let $\pi\colon \widetilde{M} \to M$ be the universal cover of M, and $\widetilde{\mathcal{T}}$ the triangulation of $\widetilde{M}$ induced from $\mathcal{T}$. Let V be the set of vertices of $\mathcal{T}$.

We take any ρ-equivariant map $F\colon \pi^{-1}(V) \to S^2_\infty$ so that for any 3-simplex σ in $\mathcal{T}$, the four points $\{F(v) \mid v \text{ is a vertex of } \sigma\}$ are distinct. The existence of such F is guaranteed by condition (2). Indeed, let $D \subset \widetilde{M}$ be a fundamental domain of M which is a union of 3-simplices of $\widetilde{\mathcal{T}}$. Let $V = \{v_1, ..., v_{|V|}\}$ and $V_i = \pi^{-1}(v_i) \cap D$. We take a point $u_i \in V_i$ for each $i \in \{1, ..., |V|\}$. Then by the ρ-equivariance, $F(V_i)$ should be determined by $F(u_i)$. Namely, if $u_i' \in V_i$ with $u_i' = \gamma \cdot u_i$ for some $\gamma \in \pi_1(M)$, then $F(u_i') = \rho(\gamma)F(u_i)$. Let e be an edge in D such that both of its vertices w_1, w_2 are in V_i for some $i \in \{1, ..., |V|\}$, i.e., $w_j = \gamma_j \cdot u_i$ for some $\gamma_j \in \pi_1(M)$, $j \in \{1, 2\}$, we see that since by condition (2), $\rho([e]) \neq 1$, there are at most two $y \in S^2_\infty$ such that $\rho([e]) \cdot y = y$. Therefore, for a generic choice of $z \in S^2_\infty$,

$$\rho(\gamma_2) \cdot z = \rho([e]) \cdot (\rho(\gamma_1) \cdot z)) \neq \rho(\gamma_1) \cdot z.$$

Since there are in total finitely many edges in D, for a generic choice of $(z_1, ..., z_{|V|}) \in (S^2_\infty)^V$, the map defined by

$$F(u_i) \doteq z_i, \quad i \in \{1, ..., |V|\}$$

and

$$F(\gamma \cdot u_i) \doteq \rho(\gamma) \cdot F(u_i), \quad \gamma \in \pi_1(M)$$

satisfies the property that for each edge e of D with vertices w_i and w_2 such that $w_1, w_2 \in V_i$ for some $i \in \{1, ..., |V|\}$,

$$F(w_2) = \rho([e]) \cdot F(w_1) \neq F(w_1).$$

Furthermore, since there are only finitely many vertices in D, for a generic choice of $(z_1, ..., z_|V|) \in (S^2_\infty)^V$, the map F satisfies that for each e in D with vertices w_1 and w_2,

$$F(w_2) \neq F(w_1).$$

Therefore, for each 3-simpliex σ in $\mathcal{T}$, the four points $\{F(v) \mid v \text{ is a vertex of } \sigma\}$ are distinct.

For any 3-simplex σ of $\mathcal{T}$, let $\widetilde{\sigma}$ in $\widetilde{\mathcal{T}}$ be a lift of σ. Then the four distinct points $\{F(v) \mid v \text{ is a vertex of } \sigma\}$ determine an ideal tetrahedron σ_∞ with them as vertices. We assign the shape parameters of σ_∞ to the corresponding normal quadrilaterals of σ, and get an assignment $Z_\rho \subset (\mathbb{C} \setminus \{0,1\})^Q$ of complex numbers to the set of normal quadrilaterals in $\mathcal{T}$. We claim that Z_ρ is a solution to the hyperbolic gluing equation.

By Proposition 2.4, we see that Z_ρ satisfies (b) of Definition 2.1. The verification of (a) is a cross-ratio calculation which is exactly the same as in [5]. We include it here for the readers' convenience. Let $e \in E$, and $\widetilde{e} \in \widetilde{\mathcal{T}}$ a lift of e with end points v and w. Let $\sigma_1, ..., \sigma_k$ be tetrahedra in $\widetilde{\mathcal{T}}$ sharing $\widetilde{e}$ as an edge in a cyclic order, and $q_i \subset \sigma_i$ be the normal quadrilateral facing $\widetilde{e}$. Let u_i and u_{i+1} be the other two vertices of σ_i so that $u_i \in \sigma_{i-1} \cap \sigma_i$. We make a convention that $u_{k+1} = u_1$. Let $\sigma_{i,\infty}$ be the ideal tetrahedron determined by vertices $F(v), F(w), F(u_i)$ and $F(u_{i+1})$, and l be the geodesic connecting $F(v)$ and $F(w)$, then $\{\sigma_{i,\infty}\}_{i=1}^k$ share l as an edge in a cyclic order. Without loss of generality, we can assume that $F(v) = 0$ and $F(w) = \infty \in S^2_\infty = \mathbb{C} \cup \{\infty\}$. Suppose z_i is the complex number assigned to q_i, i.e., the shape parameter of $\sigma_{i,\infty}$ at l, then

$$\begin{aligned}
\prod_{q \in Q} z_q^{i(q,e)} =& \prod_{i=1}^k z_i \\
=& \prod_{i=1}^k (0, \infty; F(u_i), F(u_{i+1})) \\
=& \prod_{i=1}^k \frac{F(u_i)}{F(u_{i+1})} \\
=& 1,
\end{aligned}$$

which verifies (a).

From the arbitrariness of F, we see that there are uncountably many choices of Z_ρ. ∎

We call the solutions $Z_\rho \in \mathcal{P}(M)$ constructed above *the solutions from the spinning construction.*

After we obtained the result, it was brought to our attention that similar construction had also appeared a little earlier in the work of Kashaeve, Korepanov and Martyushev [6].

3.2 Spinning construction and the proof of 2. of Theorem 1.1

According to Thurston's notes, Section 6.1, [10], any $k+1$ points $v_0, ..., v_k$, $1 \leqslant k \leqslant 3$, in $\mathbb{H}^3$ determine a *straightening map* (or *straight k-simplex*) $\sigma_{v_0,\ldots,v_k} \colon \Delta^k \to \mathbb{H}^3$, whose image is the convex hull of $v_0, ..., v_k$. Similarly, any $k+1$ points $v_0, ..., v_k$, $1 \leqslant k \leqslant 3$, in S^2_∞ determine an *ideal straightening map* (or *ideal straight k-simplex*), see Section 2.2, [5] for details. The (ideal) straightening map is natural in the following sense (see also [5] for the proof).

Proposition 3.1 *1. If Δ' is an m-face of Δ^k so that $\sigma_{v_0,\ldots,v_k}(\Delta')$ has vertices $v_{i_0}, ..., v_{i_m}$, then*

$$\sigma_{v_0,\ldots,v_k}|_{\Delta'} = \sigma_{v_{i_0},\ldots,v_{i_m}}.$$

2. *If $g \in Iso(\mathbb{H}^n)$, the group of isometries of $\mathbb{H}^3$, then*

$$g \circ \sigma_{v_0,\ldots,v_k} = \sigma_{g\cdot v_0,\ldots,g\cdot v_k}.$$

To prove 2. of Theorem 1.1, we need the following technical Lemma whose proof is contained in [5].

Lemma 3.2 *Let $\{\sigma_t\colon \Delta^k \to \mathbb{H}^3 \;\; t \in \mathbb{R}_{\geqslant 0}\}$ be a family of straight k-simplices so that the i-th vertex $v_{i,t}$ of σ_t lies in a geodesic ray l_i and $v_{i,t}$ moves toward the end point v_l^* of l_i at unit speed, i.e., $d(v_{i,0}, v_{i,t}) = t$. If $v_0^*, \ldots, v_k^*$ are pairwise distinct, then as t tends to ∞ the family $\{\sigma_t\}$ converges pointwise to an ideal straight k-simplex $\sigma_\infty\colon \Delta^k \to \overline{\mathbb{H}^3}$ whose vertices are $v_0^*, \ldots, v_k^*$.*

Proof of 2. of Theorem 1.1

We take an arbitrary ρ-equivariant map $f\colon \widetilde{M} \to \mathbb{H}^3$ and apply the following spinning construction. Let $D \subset \widetilde{M}$ be a fundamental domain of M which is a union of 3-simplices of $\widetilde{\mathcal{T}}$ with some 0-, 1- and 2-faces removed so that $\pi|_D\colon D \to M$ is one-to-one and onto, and $\mathcal{T}_D$ be the triangulation of D restricted from $\widetilde{\mathcal{T}}$. Let V_D be the set of vertices of $\mathcal{T}_D$. For each $v \in V_D$, let l_v be a geodesic in $\mathbb{H}^3$ passing through $f(v)$ $F(v) \in S^2_\infty$ as one of its end-points. We parameterize $l_v\colon (-\infty,\infty) \to \mathbb{H}^3$ so that $l_v(0) = f(v)$, $\|l_v'(t)\|_{\mathbb{H}^3} = 1$, $\forall t \in (-\infty,\infty)$, and $l_v(t) \to F(v)$ as $t \to +\infty$, and define a family of piecewise smooth ρ-equivariant maps $f_t\colon \widetilde{M} \to \mathbb{H}^3$, $t \in [0,\infty)$ as follows. Define

$$f_t(v) = exp_v(t \cdot l_v'(0)),\ \forall v \in V_D,$$

and

$$f_t(\gamma \cdot v) = \rho(\gamma) \cdot f_t(v),\ \forall \gamma \in \pi_1(M),\ v \in V_D.$$

Extend f_t to the 1-, 2- and 3-simplices of $\widetilde{\mathcal{T}}$ by straightening maps. By 1. of Proposition 3.1, f_t is well defined, and by 2. of Proposition 3.1, f_t is ρ-equivariant. From the definition of f_t, we see that for each vertex $\widetilde{v}$ of $\widetilde{\mathcal{T}}$, $f_t(\widetilde{v})$ approaches to $F(\widetilde{v}) \in S^2_\infty$, and for each face $\widetilde{\sigma}$ of $\widetilde{\mathcal{T}}$, $f_t(\widetilde{\mathcal{T}})$ lies in a totally geodesic plane.

By Lemma 3.2, $f_t\colon \widetilde{M} \to \mathbb{H}^3$ pointwise converges to a piecewise smooth ρ-equivariant map $f_\infty\colon \widetilde{M} \to \overline{\mathbb{H}^3}$ such that

1. $\forall \widetilde{v} \in \pi^{-1}(V)$, $f_\infty(\widetilde{v}) = F(\widetilde{v})$; and
2. $f_\infty(\widetilde{M} \setminus \pi^{-1}(V)) \subset \mathbb{H}^3$.

Given the solution Z_ρ to the hyperbolic gluing equation, f_∞ can be regarded as the pseudo developing map described in Section 2.3 which gives rise to the associated representation. Tautologically the map $f_\infty\colon \widetilde{M} \setminus \pi^{-1}(V) \to \mathbb{H}^3$ is ρ_{Z_ρ}-equivariant. Therefore, f_∞ is both ρ- and ρ_{Z_ρ}-equivariant, and for all $\gamma \in \pi_1(M)$ and $x \in \widetilde{M} \setminus \pi^{-1}(V)$; and we have

$$\begin{aligned}\rho_{Z_\rho}(\gamma) \cdot f_\infty(x) =& f_\infty(\gamma \cdot x)\\ =& \rho(\gamma) \cdot f_\infty(x),\end{aligned}$$

i.e., $\rho_{Z_\rho}(\gamma)|_{f_\infty(\widetilde{M}\setminus\pi^{-1}(V))} = \rho(\gamma)|_{f_\infty(\widetilde{M}\setminus\pi^{-1}(V))}$. It is clear that $f_\infty(\widetilde{M} \setminus \pi^{-1}(V))$ contains more than four points. Indeed, for and 3-simplex $\widetilde{\sigma}$ in $\widetilde{\mathcal{T}}$, the interior of the ideal tetrahedron $f_\infty(\widetilde{\sigma})$ contains an open subset of a totally geodesic plane (generically, the interior of the ideal tetrahedron $f_\infty(\widetilde{\sigma})$ is itself open in $\mathbb{H}^3$, and the only "bad" extremal case happens only if that for any $\widetilde{\sigma}$ in $\widetilde{\mathcal{T}}$, $f_\infty(\widetilde{\sigma})$ is flat). Therefore,

$$\rho_{Z_\rho}(\gamma) = \rho(\gamma) \in PSL_2(\mathbb{C}),\ \forall \gamma \in \pi_1(M),$$

i.e.,

$$\rho_{Z_\rho} = \rho\colon\ \pi_1(M) \to PSL_2(\mathbb{C}).$$

∎

Let $\mathcal{R}(M)$ be the set of representations $\rho\colon\ \pi_1(M) \to PSL_2(\mathbb{C})$. As we pointed out in the introduction, in the case that the triangulation $\mathcal{T}$ is simplicial, every $\rho \in \mathcal{R}(M)$ satisfies condition (2), and we have

Theorem 3.3 *If M is a closed, oriented 3-manifold, and $\mathcal{T}$ is a simplicial triangulation of M, then the map $Y\colon\ \mathcal{P}(M) \to \mathcal{R}(M)$ defined by $Y(Z) = \rho_Z$ is surjective.*

By 2. of Theorem 1.1, we see that our construction is the inverse construction of Yoshida's as described in Section 2.3, and we have

Theorem 3.4 *If $(M, \mathcal{T}, \rho)$ satisfies the condition of Theorem 1.1, then all the solutions Z_ρ of the hyperbolic gluing equation such that ρ_{Z_ρ} is conjugate to ρ are from the spinning construction.*

3.3 Continuous extension of the volume function and the proof of 3. of Theorem 1.1

Given a hyperbolic 3-simplex σ with vertices $v_1, ..., v_4$, the i-th face is defined to be the 2-simplex facing v_i. The dihedral angle between the i-th and j-th faces is denoted by $a_{ij}(\sigma)$. As a convention, we define $a_{ii}(\sigma) = \pi$, and call the symmetric matrix $[a_{ij}(\sigma)]_{6\times 6}$ the angle matrix of σ. It is well known that the angle matrix $[a_{ij}(\sigma)]_{6\times 6}$ determines σ up to isometry.

To prove 3. of Theorem 1.1, we need the following theorem of Luo [3].

Theorem 3.5 ***(Luo)*** *Let $X \subset \mathbb{R}^{6\times 6}$ be the space of angle matrices of all hyperbolic 3-simplices. The volume function $V\colon\ X \to \mathbb{R}$ can be extended continuously to the closure of X in $\mathbb{R}^{6\times 6}$.*

We point out that Luo's original result, Theorem 1.1 in [3], is more general than Theorem 3.5. It covers the cases of simplices in arbitrary dimensions, and in both hyperbolic and spherical geometry. See [3] for details.

Proof of 3. of Theorem 1.1

Let Z_ρ be a solution to the hyperbolic gluing equation constructed in Section 3.1. By Definition 2.2,

$$\begin{aligned} Vol(Z_\rho) &= \sum_{\sigma\in T} Vol(z_{\rho,\sigma}) \\ &= \sum_{\sigma\in\mathcal{T}_D} Vol_{\mathbb{H}^3}(f_\infty(\sigma)) \end{aligned}$$

Since $f_t\colon\ \widetilde{M} \to \mathbb{H}^3$ constructed in Section 3.2 is ρ-equivariant, $\forall t \in [0, +\infty)$, as defined by Dunfield in Section 2.5 of [1],

$$\begin{aligned} Vol(\rho) &= \int_D f_t^*(dVol_{\mathbb{H}^3}) \\ &= \sum_{\sigma\in\mathcal{T}_D} \int_\sigma f_t^*(dVol_{\mathbb{H}^3}) \\ &= \sum_{\sigma\in\mathcal{T}_D} Vol_{\mathbb{H}^3}(f_t(\sigma)), \quad \forall t \in [0, +\infty). \end{aligned}$$

For any 3-simplex $\sigma \in \mathcal{T}_D$, since $f_t|_\sigma$ pointwise converges to $f_\infty|_\sigma$, the angle matrices $[a_{ij}(f_t(\sigma))]_{6\times 6}$ converges to $[a_{ij}(f_\infty(\sigma))]_{6\times 6} \in \overline{X}$. By Theorem 3.5,

$$Vol_{\mathbb{H}^3}(f_\infty(\sigma)) = \lim_{t\to+\infty} Vol_{\mathbb{H}^3}(f_t(\sigma)).$$

Therefore,

$$\begin{aligned} Vol(Z_\rho) &= \sum_{\sigma\in\mathcal{T}_D} Vol_{\mathbb{H}^3}(f_\infty(\sigma)) \\ &= \sum_{\sigma\in\mathcal{T}_D} \lim_{t\to+\infty} Vol_{\mathbb{H}^3}(f_t(\sigma)) \\ &= \lim_{t\to+\infty} \sum_{\sigma\in\mathcal{T}_D} Vol_{\mathbb{H}^3}(f_t(\sigma)) \\ &= \lim_{t\to+\infty} Vol(\rho) \\ &= Vol(\rho) \end{aligned}$$

∎

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